Median-based Nonparametric Estimation of Returns in Mean-Downside Risk Portfolio frontier

H. Ben Salah[a,b,c], A. Gannoun[c] C. De Peretti[b], and A. Trabelsi[a]
[a] BESTMOD Laboratory, Institut Supérieur de Gestion , Tunisie
[b] ISFA .
[c] Mathematics and Modelling Institute of Montpellier, France

Résumé

The Downside Risk (DSR) model for portfolio optimisation allows to overcome the drawbacks of the classical mean-variance model concerning the asymetry of returns and the risk perception of investors. This model optimization deals with a positive definite matrix that is endogenous with respect to portfolio weights. This aspect makes the problem far more difficult to handle. For this purpose, Athayde (2001) developped a new recurcive minimization procedure that ensures the convergence to the solution. However, when a finite number of observations is available, the portfolio frontier presents some discontinuity and is not very smooth. In order to overcome that, Athayde (2003) proposed a mean kernel estimation of the returns, so as to create a smoother portfolio frontier. This technique provides an effect similar to the case in which we had continuous observations. In this paper, taking advantage on the the robustness of the median, we replace the mean estimator in Athayde's model by a nonparametric median estimator of the returns. We then give a new version of the former algorithm (of Athayde (2001, 2003)). We eventually analyse the properties of this improved portfolio frontier and apply this new method on real examples.

Keywords : Downside Risk, Kernel Method, Median, Nonparametric Estimation, Semivariance.

1 Introduction and formal framework

Optimizing asset allocation is simply defined as the process of mixing asset weights of a portfolio within the constraints of an investor's capital resources to yield the most favorable risk-return trade-off. For typical risk-averse investors, an optimal combination of investment assets that gives a lower risk and a higher return is always preferered (Markowitz, 1959). In complete market without riskless lending and borowing, a whole range of efficient asset portfolios having the stochastic dominance features could be determined, which collectively delineates a convex mean-variance frontier.

The classical mean-variance (M-V) portfolio optimization model introduced by Markowitz (1952) aims at determining the factors x_i of a given capital to be invested in each asset i belonging to a predetermined set or market so as to minimize the risk of the return of the whole portfolio, identified with its variance, while restricting the expected return of the portfolio to obtain a specified value.

More precisely, we assume that n assets are available, and we denote by μ_i the expected return of asset i , and σ_{ij} the covariance of returns of assets i and j for $i, j = 1, ..., n$. We also denote by μ the required level of return for the portfolio. The classical M-V model is

$$Min_{x_1,x_2,...,x_n} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}$$

$$\text{subject to } \sum_{i=1}^{n} x_i \mu_i = \mu \qquad \sum_{i=1}^{n} x_i = 1$$

This is a convex quadratic programming problem which can be solved by a number of efficient algorithms with a moderate computational effort even for large instance. This problem can be solved for a specific value of μ or, alternatively, for several values of μ thus generating the minimum variance set. Either way, it is important to notice, first, that the risk of the portfolio can be expressed as a function of the risk of the individual assets in the portfolio ; second, that all the variances, covariances, and expected returs of the individual assets are exogenous variables.

But, the use of variance as a risk measure is a questionable measure of risk for at least three reasons :

- it makes no distinction between gains and losses,
- it is an appropriate measure of risk only when the underlying distribution of returns is not symmetric,
- it can be applied as a risk measure only when the underlying distribution of returns is Normal.

Markowitz (1959) recognized the "asymmetrical" inefficiencies inherited in the traditional M-V models. To overcome the drawbacks of this model, he suggested to use a Downside Risk (DSR) measured by

$$\frac{1}{T} \sum_{t=1}^{T} [min(r_{pt} - B, 0)]^2$$

where r_{pt} denotes the returns of the portfolio and B a target return (the benchmark). DSR (the semivariance when $B = \mu$) is a more robust measure of asset risk that

focuses only on the risks below a target rate of return.
The DSR, is a more plausible measure of risk for several reasons : first, investors obviously do not dislike upside volatility ; they only dislike downside volatility. Second, the semivariance is more useful than the variance when the underlying distribution of returns is asymmetric and just as useful when the underlying distribution is symmetric ; in other words, the semivariance is at least as useful a measure of risk as the variance. And third, the semi-variance combines into one measure the information provided by two statistics, variance and skewness, thus making it possible to use a one-factor model to estimate required returns.
The model of optimization is the following

$$Min_{x_1,x_2,\ldots,x_n} \frac{1}{T}\sum_{t=1}^{T}[min(r_{pt}-B,0)]^2 \tag{1}$$

$$\text{subject to } \sum_{i=1}^{n} x_i\mu_i = \mu \qquad \sum_{i=1}^{n} x_i = 1,$$

But finding the portfolios with minimum semivariance is not an easy task. The major obstacle to the solution of this problem is that the semicovariance matrix is endogenous ; that is, a change in weights affects the periods in wich the portfolio underperforms the target rate of return, which in turn affects the element of the semivariance matrix.
Many approaches are suggested to estimate the portfolio semivariance. For example, Markowitz (1959) proposed the following estimator :

$$\Sigma_{pB}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \Sigma_{ijB}, \quad \text{where} \quad \Sigma_{ijB} = \frac{1}{T}\sum_{j=1}^{K}(r_{it}-B)(r_{jt}-B),$$

where periods 1 through K are periods in wich the portfolio underperforms the target return B.
This estimator has one advantage and one drawback. The advantage is that it provides an exact estimation of the portfolio semivariance, and the drawback is that the semicocariance matix is still endogenous. Many authors proposed differents ways to resolve problem (1). Among them, Hagan and Warren (1974) propose to use the the Frank-Wolf algorithm. Ang (1975) proposes to linearize the semivariance so that the optimization problem can be soved using linear programming. Harlow (1991) also considers the problem (1) and generate mean-semivariance efficient frontiers, which he compares to the mean-variance efficient frontiers. Mamoghli and Daboussi (2008) improve Harlow approach and their model permits to surmont the problem of inequality of the cosemivariance measures which occurs in the mean-semivariance model of Harlow. Markowitz et al (1993) transform the mean-semivariance problem into quadratic problem by adding fictius securities. Estrada (2008) proposes a simple and accurate heuristic approach that yields a symetric and exagenous semicovariance

matrix, which enables the determination of mean-semivariance optimal portfolios by using the well known closed-form solutions of mean-variance problems. Athayde (2003) generalises his own iterative algorithm developed in (2001) to construct a mean-downside risk portfolio frontier. The major contribution in the previous papers is to replace returns by their mean kernel estimations. The great advantage of this technique is that it provides an effect similar to the case in wich we had continuous observations. The new portfolio frontier has a smoother shape than the traditional one. In this communication, taking advantage on the the robustness of the median, we replace the mean estimators in Althayde by a nonparametric median estimators of the returns. In the next section, we exhibit these estimators and we give some elements to make them efficient

2 Nonparametric median estimation and Mean DSR estimation

The disadvantage of above mean regression is that it is sensitive to outliers and may be inappropriate in some cases, as when the distribution is multimodal or highly asymmetric. This problems can be solved by using another useful descriptive statistic which is robust to heavy-tailed error distributions and outliers : the median regression which is a more complete picture of the distribution.
Let us say that we have n assets, and denote by r_{it} the return of asset i on date t. Following the algorithm of Athayde (2003), for a fixed asset i, we replace all the observations $r_{it}, t = 1, ..., T$ by their median kernel estimators (median regression) $\hat{r}_{it}, t = 1, ..., T$ which are defined by

$$\hat{r}_{it} = \hat{F}^{-1}(\frac{1}{2}|r_{it}) \tag{2}$$

where $\hat{F}_T(.|r_{it})$ is a kernel estimator of the conditional distribution $F(.|r_{it})$ obtained by replacing, in the kernel mean estimator, r_{ij} by the indicator $1_{\{r_{ij} \leq z\}}$, so

$$\hat{F}_T(z|r_{it}) = \frac{\sum_{j=1}^{T} 1_{\{r_{ij} \leq z\}} K(\frac{r_{ij}-r_{it}}{h})}{\sum_{j=1}^{T} K(\frac{r_{ij}-r_{it}}{h})}, \tag{3}$$

we can also get directly this estimator using Koenker (2005) :

$$\hat{r}_{it} = \arg min_{z \in \mathbb{R}} \frac{\sum_{j=1}^{T} |r_{ij} - z| K(\frac{r_{ij}-r_{it}}{h})}{\sum_{j=1}^{T} K(\frac{r_{ij}-r_{it}}{h})}, \tag{4}$$

where $K(x)$ is a function that decreases with x (a probability density function). The term h (the bandwidth) is chosen in order to penalise the distance between r_{it}

and r_{ij}. It is not hard to see that the estimators $\hat{r}_{it}$ will tend to be smoother thant the original series r_{it} (see for example Silverman (1986) and Gannoun et al (2003) for details on nonparametric estimation).
The new estimation of the downside risk noted MMDSR is given, for the asset i, by

$$MMDSR = \frac{1}{T}\sum_{t=1}^{T}[min(\hat{r}_{it} - B), 0]^2. \tag{5}$$

The optimization problem (1) is now transformed to

$$Min_{x_1,x_2,\dots,x_n}\frac{1}{T}\sum_{t=1}^{T}[min(\hat{r}_{pt} - B, 0)]^2 \tag{6}$$

$$\text{subject to } \sum_{i=1}^{n} x_i\mu_i = \mu \qquad \sum_{i=1}^{n} x_i = 1,$$

where $\hat{r}_{pt}$ estimated returns of the portfolio and B a target return.

To get the solution, we follow, step by step, the algorithm of Athayde. We start with an arbitrary portfolio and, iteratively, obtain new portfolios with MMDSR smaller and smaller on every iteration. We stop titerating when if the changes in the portfolio are smaller than a fixed limit. The construction of the portfolio frontier remains the same as in Athayde (2003) but median estimators replace the mean ones. Real data, from different markets, will be used to support all points of this paper.

References

Ang, J. (1975). A note on the ESL Portfolio Selection Model. Journal of Financial and Quantitatice Analysis, 10, 849-857.
Athayde, G. (2001) Building a Mean-Downside Risk Portfolio Frontier, Developments in Forecast Combination and Portfolio Choice, John Wiley and Sons.
Athayde, G. (2003) The mean-downside risk portfolio frontier : a non-parametric approach Published in Advances in portfolio construction and implementation.
Estrada, J. (2004). Mean-Semivariance Behavior : An Alternative Behavioral Model. Journal of Emerging Market Finance, 3, 231-248.
Estrada, J. (2008). Semivariance Optimization : A Heuristic Approach.Journal of Applied Finance, 57-72.
Gannoun, A, Saracco, J. and Yu, K. (2003). Nonparametric time series prediction by conditional median and quantiles. Journal of statistical Planning and inference, 117, 207-223.
Hogan, W. and Warren, J. (1974). Computation of the efficient Boundary in the ES Portfolio selection. Journal of Financial and Quantitatice Analysis, 9, 1-11.
Harlow, V. (1991). Asset allocation in a downside risk framwork. Financial Analyst Journal, 47, 28-40.

Mamoghli, C. and Daboussi, S. (2008). Optimisation de portefeuille downside risk . Social Science Research Network. 23 pages.
Markowitz, H. (1952) Portfolio Selection, Journal of Finance, 7, 77–91.
Markowitz, H.M. (1959). Portfolio Selection : Efficient Diversification of Investments. New York : John Wiley & Sons.
Markowitz, H.M. (1991). Portfolio Selection : Efficient Diversification of Investments. Basil Blakwell.
Markowitz, M., Peter,T., Ganlin, X. and Yuji,Y. (1993) Computation of mean-semivariance Efficient sets by the Critical Line Algorithm. Annals of Operationss Research, 45, 307-317.
Silverman, B.W. (1986) Density Estimation for Statistics and Data Analysis, Chapman and Hall, New York.