Median-based Nonparametric Estimation of Returns in Mean-Downside Risk Portfolio frontier

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Résumé

The Downside Risk (DSR) model for portfolio optimisation allows to overcome the drawbacks of the classical mean-variance model concerning the asymmetry of returns and the risk perception of investors. This model optimization deals with a positive definite matrix that is endogenous with respect to portfolio weights. This aspect makes the problem far more difficult to handle. For this purpose, Athayde (2001) developed a new recursive minimization procedure that ensures the convergence to the solution. However, when a finite number of observations is available, the portfolio frontier presents some discontinuity and is not very smooth. In order to overcome that, Athayde (2003) proposed a mean kernel estimation of the returns, so as to create a smoother portfolio frontier. This technique provides an effect similar to the case in which we had continuous observations. In this paper, taking advantage of the robustness of the median, we replace the mean estimator in Athayde’s model by a nonparametric median estimator of the returns. We then give a new version of the former algorithm (of Athayde (2001, 2003)). We eventually analyse the properties of this improved portfolio frontier and apply this new method on real examples.

Keywords: Downside Risk, Kernel Method, Median, Nonparametric Estimation, Semivariance.

1 Introduction and formal framework

Optimizing asset allocation is simply defined as the process of mixing asset weights of a portfolio within the constraints of an investor’s capital resources to yield the
most favorable risk-return trade-off. For typical risk-averse investors, an optimal combination of investment assets that gives a lower risk and a higher return is always preferred (Markowitz, 1959). In complete market without riskless lending and borrowing, a whole range of efficient asset portfolios having the stochastic dominance features could be determined, which collectively delineates a convex mean-variance frontier.

The classical mean-variance (M-V) portfolio optimization model introduced by Markowitz (1952) aims at determining the factors $x_i$ of a given capital to be invested in each asset $i$ belonging to a predetermined set or market so as to minimize the risk of the return of the whole portfolio, identified with its variance, while restricting the expected return of the portfolio to obtain a specified value. More precisely, we assume that $n$ assets are available, and we denote by $\mu_i$ the expected return of asset $i$, and $\sigma_{ij}$ the covariance of returns of assets $i$ and $j$ for $i, j = 1, ..., n$. We also denote by $\mu$ the required level of return for the portfolio. The classical M-V model is

$$
\text{Min } x_1, x_2, ..., x_n \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}
$$

subject to

$$
\sum_{i=1}^{n} x_i \mu_i = \mu \quad \sum_{i=1}^{n} x_i = 1
$$

This is a convex quadratic programming problem which can be solved by a number of efficient algorithms with a moderate computational effort even for large instance. This problem can be solved for a specific value of $\mu$ or, alternatively, for several values of $\mu$ thus generating the minimum variance set. Either way, it is important to notice, first, that the risk of the portfolio can be expressed as a function of the risk of the individual assets in the portfolio; second, that all the variances, covariances, and expected returns of the individual assets are exogenous variables.

But, the use of variance as a risk measure is a questionable measure of risk for at least three reasons:

- it makes no distinction between gains and losses,
- it is an appropriate measure of risk only when the underlying distribution of returns is not symmetric,
- it can be applied as a risk measure only when the underlying distribution of returns is Normal.

Markowitz (1959) recognized the "asymmetrical" inefficiencies inherited in the traditional M-V models. To overcome the drawbacks of this model, he suggested to use a Downside Risk (DSR) measured by

$$
\frac{1}{T} \sum_{t=1}^{T} \left[ \text{min}(r_{pt} - B, 0) \right]^2
$$

where $r_{pt}$ denotes the returns of the portfolio and $B$ a target return (the benchmark). DSR (the semivariance when $B = \mu$) is a more robust measure of asset risk that
focuses only on the risks below a target rate of return.

The DSR, is a more plausible measure of risk for several reasons: first, investors obviously do not dislike upside volatility; they only dislike downside volatility. Second, the semivariance is more useful than the variance when the underlying distribution of returns is asymmetric and just as useful when the underlying distribution is symmetric; in other words, the semivariance is at least as useful a measure of risk as the variance. And third, the semi-variance combines into one measure the information provided by two statistics, variance and skewness, thus making it possible to use a one-factor model to estimate required returns.

The model of optimization is the following

\[
\min_{x_1, x_2, \ldots, x_n} \frac{1}{T} \sum_{t=1}^{T} \left[ \min(r_{pt} - B, 0) \right]^2
\]

subject to \( \sum_{i=1}^{n} x_i \mu_i = \mu \quad \sum_{i=1}^{n} x_i = 1 \),

But finding the portfolios with minimum semivariance is not an easy task. The major obstacle to the solution of this problem is that the semicovariance matrix is endogenous; that is, a change in weights affects the periods in which the portfolio underperforms the target rate of return, which in turn affects the element of the semivariance matrix.

Many approaches are suggested to estimate the portfolio semivariance. For example, Markowitz (1959) proposed the following estimator:

\[
\Sigma_{pB}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_ix_j \Sigma_{ijB}, \text{ where } \Sigma_{ijB} = \frac{1}{K} \sum_{j=1}^{K} (r_{it} - B)(r_{jt} - B),
\]

where periods 1 through \( K \) are periods in which the portfolio underperforms the target return \( B \).

This estimator has one advantage and one drawback. The advantage is that it provides an exact estimation of the portfolio semivariance, and the drawback is that the semicovariance matrix is still endogenous. Many authors proposed different ways to resolve problem (1). Among them, Hagan and Warren (1974) propose to use the Frank-Wolf algorithm. Ang (1975) proposes to linearize the semivariance so that the optimization problem can be solved using linear programming. Harlow (1991) also considers the problem (1) and generate mean-semivariance efficient frontiers, which he compares to the mean-variance efficient frontiers. Mamogli and Daboussi (2008) improve Harlow approach and their model permits to surmount the problem of inequality of the cosemivariance measures which occurs in the mean-semivariance model of Harlow. Markowitz et al (1993) transform the mean-semivariance problem into quadratic problem by adding fictitious securities. Estrada (2008) proposes a simple and accurate heuristic approach that yields a symmetric and exogenous semicovariance...
matrix, which enables the determination of mean-semivariance optimal portfolios by using the well known closed-form solutions of mean-variance problems. Athayde (2003) generalises his own iterative algorithm developed in (2001) to construct a mean-downside risk portfolio frontier. The major contribution in the previous papers is to replace returns by their mean kernel estimations. The great advantage of this technique is that it provides an effect similar to the case in which we had continuous observations. The new portfolio frontier has a smoother shape than the traditional one. In this communication, taking advantage on the robustness of the median, we replace the mean estimators in Althayde by a nonparametric median estimators of the returns. In the next section, we exhibit these estimators and we give some elements to make them efficient.

2 Nonparametric median estimation and Mean DSR estimation

The disadvantage of above mean regression is that it is sensitive to outliers and may be inappropriate in some cases, as when the distribution is multimodal or highly asymmetric. This problems can be solved by using another useful descriptive statistic which is robust to heavy-tailed error distributions and outliers: the median regression which is a more complete picture of the distribution.

Let us say that we have $n$ assets, and denote by $r_{it}$ the return of asset $i$ on date $t$. Following the algorithm of Athayde (2003), for a fixed asset $i$, we replace all the observations $r_{it}, t = 1, ..., T$ by their median kernel estimators (median regression) $\hat{r}_{it}, t = 1, ..., T$ which are defined by

$$\hat{r}_{it} = \hat{F}^{-1}(\frac{1}{2} | r_{it})$$ (2)

where $\hat{F}_T(\cdot | r_{it})$ is a kernel estimator of the conditional distribution $F(\cdot | r_{it})$ obtained by replacing, in the kernel mean estimator, $r_{ij}$ by the indicator $1\{r_{ij} \leq z\}$, so

$$\hat{F}_T(z | r_{it}) = \frac{\sum_{j=1}^{T} 1\{r_{ij} \leq z\} K(\frac{r_{ij} - r_{it}}{h})}{\sum_{j=1}^{T} K(\frac{r_{ij} - r_{it}}{h})},$$ (3)

we can also get directly this estimator using Koenker (2005):

$$\hat{r}_{it} = \arg\min_{z \in \mathbb{R}} \sum_{j=1}^{T} |r_{ij} - z| K(\frac{r_{ij} - r_{it}}{h}),$$ (4)

where $K(x)$ is a function that decreases with $x$ (a probability density function). The term $h$ (the bandwidth) is chosen in order to penalise the distance between $r_{it}$
and \( r_{ij} \). It is not hard to see that the estimators \( \hat{r}_{it} \) will tend to be smoother than the original series \( r_{it} \) (see for example Silverman (1986) and Gannoun et al (2003) for details on nonparametric estimation).

The new estimation of the downside risk noted MMDSR is given, for the asset \( i \), by

\[
MMDSR = \frac{1}{T} \sum_{t=1}^{T} \left[ \min(\hat{r}_{it} - B, 0) \right]^2.
\]  

The optimization problem (1) is now transformed to

\[
\text{Min}_{x_{1},x_{2},...,x_{n}} \frac{1}{T} \sum_{t=1}^{T} \left[ \min(\hat{r}_{pt} - B, 0) \right]^2
\]

subject to \( \sum_{i=1}^{n} x_{i} \mu_{i} = \mu \), \( \sum_{i=1}^{n} x_{i} = 1 \),

where \( \hat{r}_{pt} \) estimated returns of the portfolio and \( B \) a target return.

To get the solution, we follow, step by step, the algorithm of Athayde. We start with an arbitrary portfolio and, iteratively, obtain new portfolios with MMDSR smaller and smaller on every iteration. We stop iterating when if the changes in the portfolio are smaller than a fixed limit. The construction of the portfolio frontier remains the same as in Athayde (2003) but median estimators replace the mean ones. Real data, from different markets, will be used to support all points of this paper.

References


