On certain transformations of Archimedean copulas: Application to the non-parametric estimation of their generators

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Résumé. Nous étudions l'impact de certaines transformations dans la classe des copules Archimédiennes. Nous donnons quelques conditions d'admissibilité pour ces transformations, nous définissons des classes d'équivalence pour les transformations et les générateurs de copules Archimédiennes. Nous étendons la composition r-fois de la section diagonale de copule, de $r \in \mathbb{N}$ à $r \in \mathbb{R}$. Cette extension, couplée avec des résultats sur les classes d'équivalence, nous donne de nouvelles expressions pour les transformations et les générateurs. Des estimateurs pour les transformations et les générateurs découlant directement de ces expressions sont proposés et leur convergence est étudiée. Nous fournissons des bandes de confiance pour les générateurs estimés. Des illustrations numériques montrent la performance empirique de ces estimateurs.

Mots-clés. Transformations de copules Archimédiennes, estimation non paramétrique, dépendance asymptotique.

Abstract. We study the impact of certain transformations within the class of Archimedean copulas. We give some admissibility conditions for these transformations, and define some equivalence classes for both transformations and generators of Archimedean copulas. We extend the r-fold composition of the diagonal section of a copula, from $r \in \mathbb{N}$ to $r \in \mathbb{R}$. This extension, coupled with results on equivalence classes, gives us new expressions of transformations and generators. Estimators deriving directly from these expressions are proposed and their convergence is investigated. We provide confidence bands for the estimated generators. Numerical illustrations show the empirical performance of these estimators.

Keywords. Transformations of Archimedean copulas, self-nested diagonal, non-parametric estimation, tail dependence.

1 The model

We consider here a particular transformation of a copula, using a function T and leading to the definition of a transformed copula $\widetilde{C}$ of an initial copula C_0,

$$\widetilde{C}(u_1, \ldots, u_d) = T \circ C_0(T^{-1}(u_1), \ldots, T^{-1}(u_d)), \quad \text{for } u_1, \ldots, u_d \in [0, 1], \tag{1}$$

see also Di Bernardino, and Rullière (2013a). The function $T : [0, 1] \to [0, 1]$ is a continuous and increasing function on the interval $[0, 1]$, with $T(0) = 0,\ T(1) = 1$, with supplementary assumptions that will be chosen to guarantee that $\widetilde{C}$ is also a copula, detailed hereafter.

In the following, we define the notion of *self-nested diagonal*.

Definition 1.1 (Discrete self-nested diagonal). *Consider a $d-$dimensional copula C such that for all $u \in [0,1]$, $\delta_1(u) := C(u,\dots,u)$ is a strictly increasing function of u. The respective discrete self-nested diagonal of C of order k and $-k$ are the functions δ_k and δ_{-k} such that for all $u \in [0,1]$, for all $k \in \mathbb{N}$,*

$$\left\{ \begin{array}{lcll} \delta_k(u) & = & \delta_1 \circ \ldots \circ \delta_1(u), & (k \text{ times}) \\ \delta_{-k}(u) & = & \delta_{-1} \circ \ldots \circ \delta_{-1}(u), & (k \text{ times}) \\ \delta_0(u) & = & u. & \end{array} \right. \tag{2}$$

where δ_{-1} is the inverse function of δ_1, so that $\delta_1 \circ \delta_{-1}$ is the identity function.

The following definition aims at defining the r-fold composition of the diagonal section δ_1 of the copula when $r \in \mathbb{R}$ is not a relative integer.

Definition 1.2 (Self-nested diagonals). *Functions of a family $\{\delta_r\}_{r\in\mathbb{R}}$ are called (extended) self-nested diagonals of a copula C, if $\delta_k(u)$ is the discrete self-nested diagonal of C of order k for all $k \in \mathbb{Z}$, as in Definition 1.1, and if furthermore*

$$\delta_{r_1+r_2}(u) \quad = \quad \delta_{r_1} \circ \delta_{r_2}(u), \quad \forall\, r_1, r_2 \in \mathbb{R}, \forall\, u \in [0,1].$$

We introduce below the notion of *self-nested diagonal* of an Archimedean copula.

Lemma 1.1 (Self-nested diagonal of an Archimedean copula). *If C is an Archimedean copula associated with a generator ϕ, then a family of self-nested diagonal of C is defined at each order $r \in \mathbb{R}$ by*

$$\delta_r(x) \quad = \quad \phi(d^r \cdot \phi^{-1}(x)), \qquad \text{for } \; x \in (0,1), \; r \in \mathbb{R}.$$

1.1 New expressions of transformations and generators using self-nested diagonals

The following result provides an expression for the transformations T of Archimedean copulas in terms of the self-nested diagonals.

Proposition 1.1 (Transformation T using self-nested diagonals). *Consider an Archimedean copula C_0 and a transformed copula $\widetilde{C}$, such that $\widetilde{C}(u_1,...,u_d) = T \circ C_0(T^{-1}(u_1),\dots,T^{-1}(u_d))$. Consider the two associated families of self-nested diagonals δ_r and $\widetilde{\delta}_r$, $r \in \mathbb{R}$ as defined in Lemma 1.1. If $T(x_0) = y_0$, then T is such that $T(0) = 0$, $T(1) = 1$ and for all $x \in (0,1)$,*

$$T(x) = \widetilde{\delta}_{r(x)}(y_0),$$
$$\text{with } r(x) \text{ such that } \delta_{r(x)}(x_0) = x,$$

where $(x_0, y_0) \in (0,1)^2$ can be arbitrarily chosen. In the case where C_0 is the independence copula,

$$r(x) = \frac{1}{\ln d} \ln\left(\frac{-\ln x}{-\ln x_0}\right).$$

Proposition 1.2 (Transformed generator $\widetilde{\phi}$ using self-nested diagonals). *Consider an Archimedean copula $\widetilde{C}$ and the associated family of self-nested diagonals $\widetilde{\delta}_r$, for $r \in \mathbb{R}$, as defined in Lemma 1.1. Assume that the copula $\widetilde{C}$ is reachable by transforming an initial Archimedean copula C_0 with a strict generator ϕ. Then the generator $\widetilde{\phi}$ of $\widetilde{C}$ is such that, for all $t \in \mathbb{R}^+ \setminus \{0\}$,*

$$\widetilde{\phi}(t) = \widetilde{\delta}_{\rho(t)}(y_0)\,,$$
$$\text{with } \rho(t) = \tfrac{1}{\ln d} \ln\left(\tfrac{t}{\phi^{-1}(x_0)}\right).$$

where $(x_0, y_0) \in (0,1)^2$ can be arbitrarily chosen. This expression does only depend on ϕ via the constant $t_0 = \phi^{-1}(x_0)$. In particular, choosing an initial copula C_0 and constants (x_0, y_0) in $(0,1)^2$ can be simply reduced to the choice of $(t_0, \varphi_0) \in \mathbb{R}^+ \setminus \{0\} \times (0,1)$ such that $\widetilde{\phi}(t_0) = \varphi_0$, with $t_0 = \phi^{-1}(x_0)$ and $\varphi_0 = y_0$.

2 Non-parametric estimation

Using (empirical) estimator of δ_1 and δ_{-1} one can easily build estimators of a whole family of self-nested diagonals $\{\delta_r\}_{r\in\mathbb{R}}$ (see for instance Deheuvels, 1979).

Definition 2.1 (Estimation of self-nested diagonals)**.** *Consider a copula C as in Definition 1.1. Let $\widehat{\delta}_1$ be an estimator of δ_1, and $\widehat{\delta}_{-1}$ be an estimator of the inverse function δ_{-1}.*
Estimators of δ_k and δ_{-k} can be obtained for any $k \in \mathbb{N} \setminus \{0\}$ by setting

$$\begin{cases} \widehat{\delta}_k(u) &= \widehat{\delta}_1 \circ \ldots \circ \widehat{\delta}_1(u), & (k \text{ times}) \\ \widehat{\delta}_{-k}(u) &= \widehat{\delta}_{-1} \circ \ldots \circ \widehat{\delta}_{-1}(u), & (k \text{ times}) \\ \widehat{\delta}_0(u) &= u. \end{cases} \tag{3}$$

However, estimating the whole functions $\widetilde{\phi}$ and T using some pre-calculations may avoid repeating some steps. Algorithms 1 and 2 show how to store some quantities in order to get readily calculable estimators of T and $\widetilde{\phi}$.

Algorithm 1 Procedure for non-parametric estimation of a transformation T

Input parameters
Choose x_0, y_0, arbitrary values in $(0,1)$, *e.g.* $x_0 = y_0 = \mathrm{e}^{-1}$
Choose $k_{\min}$, $k_{\max}$ in $\mathbb{Z}$, pre-calculation range, *e.g.* $k_{\min} = -20$, $k_{\max} = 20$.
Choose C_0 and ϕ, initial Archimedean copula and its associated generator, *e.g.* $\phi(t) = \mathrm{e}^{-t}$ *(independence).*
Choose $\widehat{\delta}_1$, an estimator for δ_1, and its inverse $\widehat{\delta}_{-1}$, *e.g. the one of Deheuvels (1979)*.

Eventual pre-calculations
For $k \in \{k_{\min}, \ldots, k_{\max}\}$, store $\widehat{\delta}_k(y_0)$ obtained by Equation (3),

Estimation
Define the function $\widehat{\delta}_r(y_0)$ for any $r \in \mathbb{R}$,
using previous stored values when $r \in [k_{\min}, k_{\max}]$, or using Equation (3) otherwise.

Get $\widehat{T}(x)$ for any $x \in [0,1]$, by Proposition 1.1

Algorithm 2 Procedure for non-parametric estimation of generators of Archimedean copulas

Input parameters
Choose t_0, φ_0, arbitrary values in $\mathbb{R}^+ \setminus \{0\} \times (0,1)$, *e.g.* $t_0 = 1, \varphi_0 = \mathrm{e}^{-1}$
Choose $k_{\min}$, $k_{\max}$ in $\mathbb{Z}$, pre-calculation range, *e.g.* $k_{\min} = -20$, $k_{\max} = 20$.
Choose z, an interpolation function, *e.g.* $z(x) = \exp(-x)$.
Choose $\widehat{\delta}_1$, an estimator for δ_1, and its inverse $\widehat{\delta}_{-1}$, *e.g. the one of Deheuvels (1979)*.

Eventual pre-calculations
For $k \in \{k_{\min}, \ldots, k_{\max}\}$, store $\widehat{\delta}_k(\varphi_0)$ obtained by Equation (3),

Estimation
Define the function $\widehat{\delta}_r(\varphi_0)$ for any $r \in \mathbb{R}$,
using previous stored values when $r \in [k_{\min}, k_{\max}]$, or using Equation (3) otherwise.

Get $\widehat{\phi}(t)$ for any $t \in \mathbb{R}^+$, by Proposition 1.2.

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3 Numerical illustrations

In this section we provide some numerical illustrations of the proposed non-parametric estimation procedure for the transformation T and the generator $\widetilde{\phi}$. Furthermore, we estimate the diagonal of the copula $\delta_1(u) := C(u, \ldots, u)$ and its inverse function δ_{-1} using the consistent empirical copula $\widehat{C}$.

Estimation of a self-nested diagonal: In Figure 1 we provide an illustration of the estimation of self-nested diagonals (see Definition 2.1): we generate a sample of size $n = 1500$ from a Clayton copula with parameter $\theta = 6$ (left) or a Gumbel copula with parameter $\theta = 3$ (right). We consider $k = -3, -2, -1, 0, 1, 2, 3$ and we estimate the self-nested diagonal $\widehat{\delta}_k(u)$, for $u \in [0, 1]$.

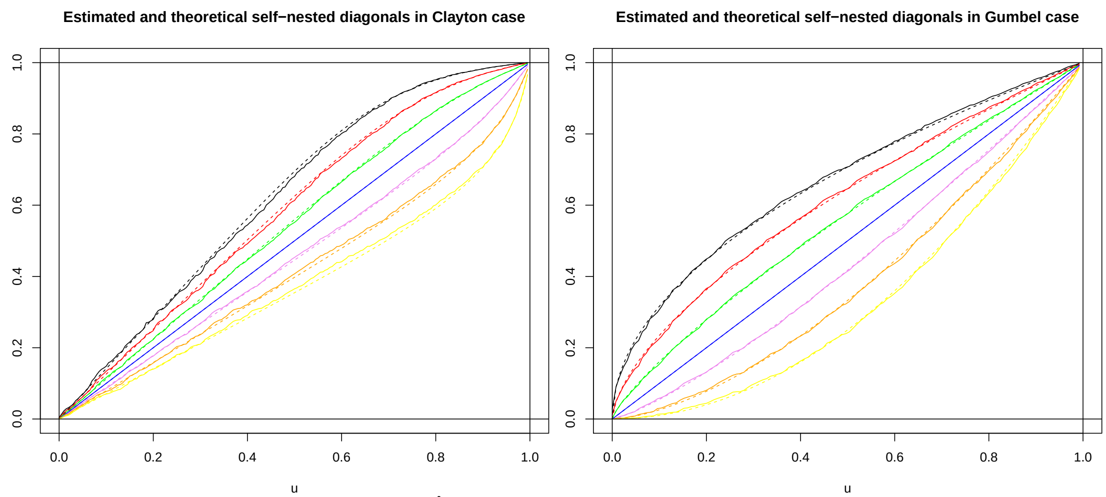

Figure 1: Estimation of self-nested diagonal $\widehat{\delta}_k(u)$ as in Definition 2.1 in the Clayton-case with parameter $\theta = 6$ (left), or in the Gumbel-case with parameter $\theta = 3$ (right) for $k = -3, -2, -1, 0, 1, 2, 3$. The estimated $\widehat{\delta}_k(u)$ are represented using full lines, the theoretical one's using dotted lines. The black upper curve corresponds to $k = -3$, the yellow lower curve to $k = 3$.

Estimation of the transformation T: In Figure 2 we have drawn the non-parametric estimation for the transformation T starting from the independence initial copula C_0, i.e. $\widehat{T}(x) = \widehat{\delta}_{r(x)}(y_0)$, with $r(x) = \frac{1}{\ln d} \ln\left(\frac{-\ln x}{-\ln x_0}\right)$. We have chosen here $x_0 = y_0 = 0.5$. We generate two samples of size $n = 1500$ from a Clayton (Figure 2, left) and a Gumbel (Figure 2, right) copulas for different Kendall's τ. In both cases we take as interpolation function $z(x) = \exp(-x)$, $x \in (0, 1]$.

Lambda function: Following the same methodology as Genest et al. (2011), we have estimated the λ function, for our estimator and for the estimator of Genest et al. (2011). For our estimator, an approximation of the derivative by finite differences leads to following estimator of λ, for a small value of h, $u \in (0, 1)$, $h < \widehat{\phi}^{-1}(u)$:

$$\widehat{\lambda}(u) = \widehat{\phi}^{-1}(u) \cdot \frac{\widehat{\phi}(\widehat{\phi}^{-1}(u) + h) - \widehat{\phi}(\widehat{\phi}^{-1}(u) - h)}{2h}. \quad (4)$$

In Figure 3, we have estimated the λ function, we get $\widehat{\lambda}(u)$ and $\widehat{\lambda}^*(u)$ for our estimator, and $\widehat{\lambda}_G(u)$ for the estimator by Genest et al. (2011). The chosen parameter setting in Figure 3 is exactly as in Figure

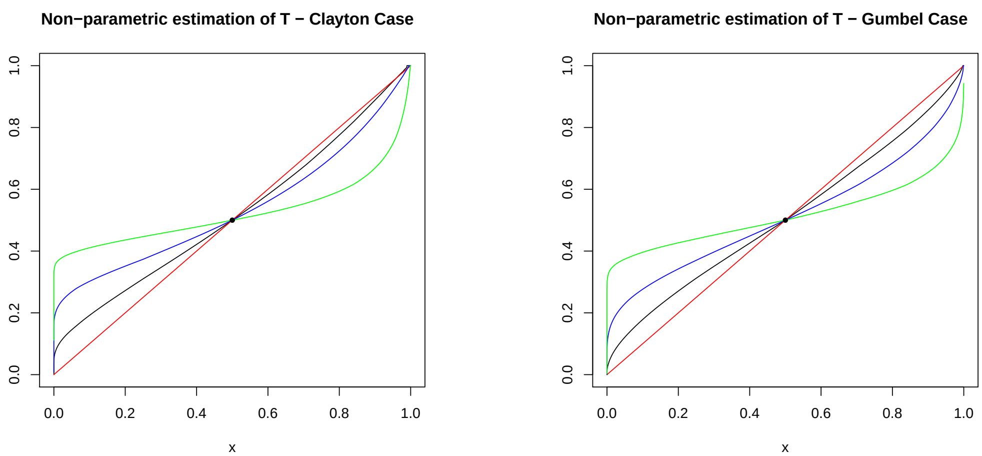

Figure 2: Non-parametric $\widehat{T}(x)$ estimated on a sample of size $n = 1500$. Bivariate Clayton-case (left) and bivariate Gumbel-case (right) with Kendall's $\tau = 0.25$ (black lines), $\tau = 0.5$ (blue lines), $\tau = 0.75$ (green lines). The red line represents the bisectrix of the quadrant. Each transformation $T(x)$ is passing through the point $(0.5, 0.5)$ (black point).

2 in Genest et al. (2011). The results show that, empirically on this data-set, all these estimators are very close.

NOTE: All details can be founded in Di Bernardino and Rullière (2013b).

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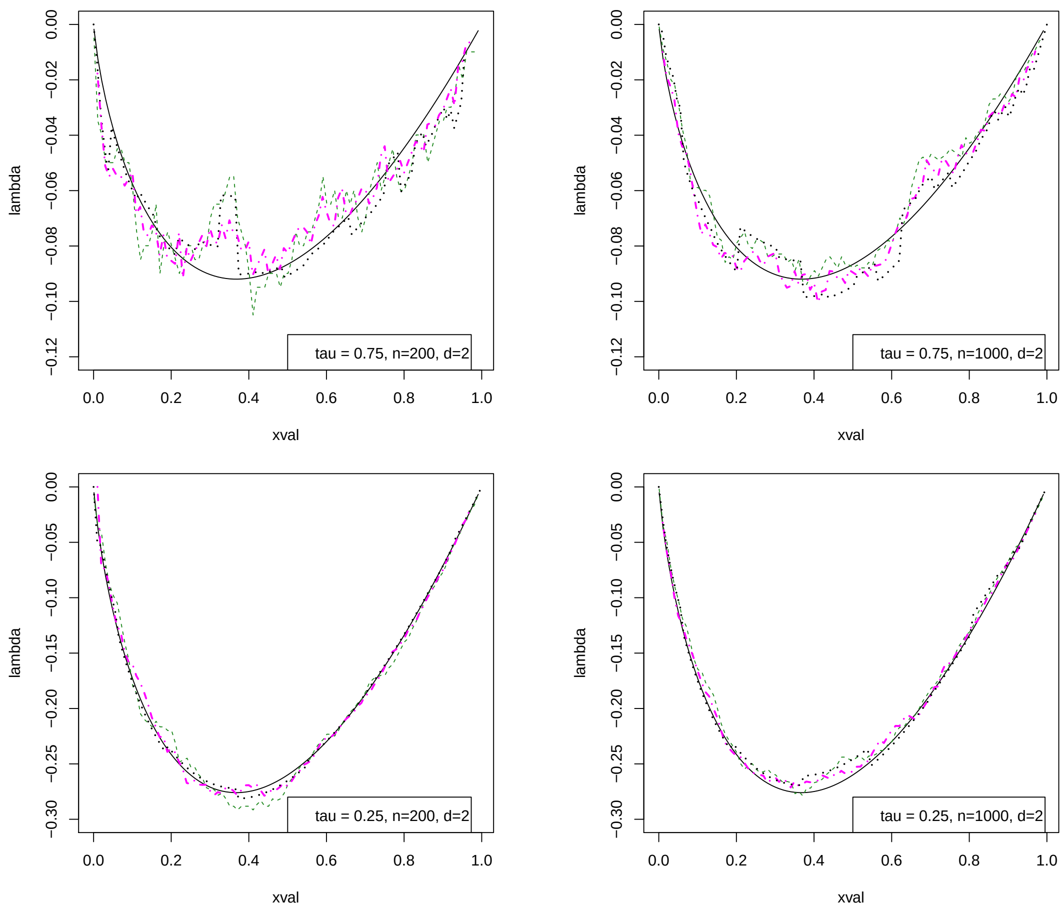

Figure 3: Estimation of λ function. Black: theoretical λ function. Dark green dashed line: $\widehat{\lambda}_G(u)$ (estimator proposed by Genest et al. (2011)), Black dotted line: $\widehat{\lambda}(u)$ (i.e., our estimator using Equation (4)). Parameters setting : $n = 200$, (right column) and $n = 1000$ (left column), $d = 2$. Kendall's tau parameter is $\tau = 0.75$ ($\theta = 4$) (upper row) and $\tau = 0.25$ ($\theta = 1.333$) (bottom row).